Solution (#117) The circle $|z-1|=1$ has a natural parametrization

$$z = 1 + \text{cis}\alpha \qquad 0 \leqslant \alpha < 2\pi.$$

As noted in #26 we have

$$z = 2\cos\left(\frac{\alpha}{2}\right)\text{cis}\left(\frac{\alpha}{2}\right).$$

So

$$z^2 = 4\cos^2\left(\frac{\alpha}{2}\right)\text{cis}\left(\alpha\right)$$

Let $r = \left|z^2\right|$ and $\theta = \arg(z^2)$. Taking arguments then

$$\theta = \alpha \qquad \text{and} \qquad r = 4\cos^2\left(\frac{\alpha}{2}\right) = 2 + 2\cos\alpha = 2 + 2\cos\theta$$

as required.

Alternative Solution: Under the map $z \mapsto z^2$ the point $x+yi$ maps to $x+yi$ where $X = x^2 - y^2$ and $Y = 2xy$. Solving for x^2 and y^2 in terms of X and Y we obtain

$$x^2 - \frac{Y^2}{4x^2} = X \quad \Longrightarrow \quad x^2 = \frac{1}{2}\left(\sqrt{X^2+Y^2} + X\right) \text{ and } y^2 = \frac{1}{2}\left(\sqrt{X^2+Y^2} - X\right).$$

In terms of polar co-ordinates r, θ the image point (X, Y) we have

$$x^2 = \frac{r}{2}\left(1+\cos\theta\right), \qquad y^2 = \frac{r}{2}\left(1-\cos\theta\right). \tag{8.8}$$

The equation of the original circle is

$$(x-1)^2 + y^2 = 1 \text{ or equivalently } x^2 - 2x + y^2 = 0.$$

Substituting in our expressions (8.8) this becomes the following constraint on r, θ:

$$\begin{aligned} \frac{r}{2}\left(1+\cos\theta\right) - \sqrt{2r\left(1+\cos\theta\right)} + \frac{r}{2}\left(1-\cos\theta\right) &= 0, \\ \Longrightarrow r &= \sqrt{2r\left(1+\cos\theta\right)}, \\ \Longrightarrow r &= 2\left(1+\cos\theta\right). \end{aligned}$$