Solution (#127) Let a, b be complex numbers and let $0 < k < 1$.

(i) From Apollonius' theorem we know that $|z - a| = k|z - b|$ represents a circle with centre c and radius r where

$$c = \frac{k^2 b - a}{k^2 - 1}, \qquad r = \frac{k|a - b|}{|k^2 - 1|}. \tag{8.10}$$

Note that

$$\begin{aligned} |c - a| &= \left|\frac{k^2 b - a}{k^2 - 1} - a\right| = \frac{k^2|b - a|}{|k^2 - 1|} = kr < r \\ |c - b| &= \left|\frac{k^2 b - a}{k^2 - 1} - b\right| = \frac{|b - a|}{|k^2 - 1|} = \frac{r}{k} > r \end{aligned}$$

and so a lies inside the circle and b outside it. Further we've shown $|c - a||c - b| = r^2$. Finally we see that

$$\frac{b - c}{a - c} = \left(\frac{a - b}{k^2 - 1}\right) / \left(\frac{k^2(a - b)}{k^2 - 1}\right) = \frac{1}{k^2}$$

is real and hence a, b, c are collinear (see #136 for further details if necessary).

(ii) Suppose now that we are given a circle C, and any point a within the circle other than the centre. By an appropriate choice of co-ordinates and unit length we can assume, without loss of generality, that C has centre 0 and radius 1. From part (i) above, we should choose b to be collinear with 0 and a and such that $|a||b| = 1$ – that is, we choose

$$b = \frac{a}{|a|^2} = \frac{1}{\bar{a}}$$

as the inverse point of a in C. We further set $k = |a|$ as we need $b/a = 1/k^2$. Then $|z - a| = k|z - b|$ represents a circle with centre c and radius r given by

$$\begin{aligned} c &= \frac{k^2 b - a}{k^2 - 1} = \frac{|a|^2/\bar{a} - a}{|a|^2 - 1} = \frac{(a\bar{a})/\bar{a} - a}{|a|^2 - 1} = 0, \\ r &= \frac{k|a - b|}{|k^2 - 1|} = \frac{|a|\left|a - \bar{a}^{-1}\right|}{1 - |a|^2} = \frac{|a|\left(1 - |a|^2\right)}{|\bar{a}|\left(1 - |a|^2\right)} = 1. \end{aligned}$$

So with the given choices of b and k we have represented C as a circle of Apollonius. Note in the above that our choices were in fact imposed on us by the various requirements, so that b and k are in fact unique choices.