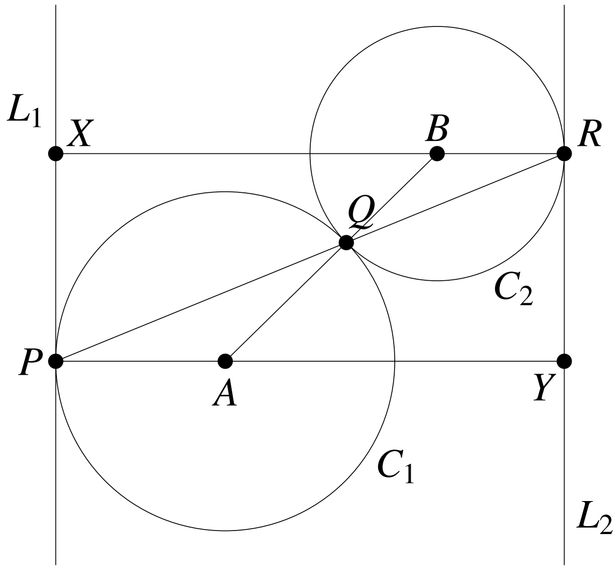
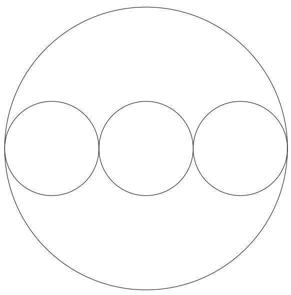
Solution (#149) Let A denote the centre of C_1 and B denote the centre of C_2 as in the diagram below.



As C_1 and C_2 are tangential at Q then AQB is a line that makes equal angles with the parallel lines XBR and PAY. So QBR and QAP are equal angles, in isosceles triangles, and hence RQB and PQA are also equal angles. This means that P, Q, R are collinear.

If we now look at the second diagram and take the origin to be the intersection of C_3 and C_4 we see, by #148, that $z \mapsto 1/z$ transforms C_3 and C_4 transform into two lines L_2 and L_1. As C_3 and C_4 meet only in one point it follows that L_1 and L_2 are parallel. We also see from #148 that C_1 and C_2 transform into two circles. The tangencies that we had in the second diagram mean that L_1 is tangential to C_1 which is tangential to C_2 which is tangential to L_2. That is, we have a version of the first diagram. The points P, Q, R respectively correspond to the other tangencies C_1 meets C_4, C_1 meets C_2 and C_2 meets C_3 in the second diagram. As P, Q, R lie on a line that does not go through the origin of the first diagram, then the line PQR resulted from transforming a circle in the second diagram that goes through its origin. As the origin of the second diagram is where C_3 meets C_4 then all four tangential points lie on this circle.

Remark: There are some subtleties to this result. It is important in the second diagram that the circles are external to one another. The result is clearly not generally true if three of the circles lie inside the fourth, for example as in the diagram below where the tangencies are collinear rather than concyclic.



If one transforms the first diagram under the map $z \mapsto 1/z$ then the resulting diagram depends on the choice of origin. If the origin is not on any of L_1, L_2, C_1, C_2 then four circles will result. If the origin is taken to be between the parallel lines and in the regions above or below C_1 and C_2 then an arrangement much like the second diagram in the exercise results with four circles externally tangential to one another. If instead one chooses the origin to be inside C_1 or C_2, to the left of L_1 or to the right of L_2, then we would have an arrangement where three circles now sat inside a fourth. If the origin was taken to be on the line PQR then the four tangencies would actually be collinear, but otherwise the tangencies would still be concyclic.

A more general version of the result would be to say that if four circles C_1, C_2, C_3, C_4 are tangential as described (whether or not externally arranged) then the tangencies lie on a circline.