Solution (#163) As p is prime then all the pth roots of unity, other than 1, are primitive. It follows that

$$\Phi_p(x) = \prod_{\substack{\omega \text{ primitive} \\ p\text{th root}}} (x-\omega) = (x-1)^{-1} \prod_{\text{all } p\text{th roots}} (x-\omega) = \frac{x^p-1}{x-1} = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

Now k is coprime with p^n if and only if it is coprime with p. So

$$\begin{aligned} \Phi_{p^n}(x) &= \prod_{k \text{ coprime with } p^n} (x - \operatorname{cis}(2k\pi/p^n) = \prod_{k \text{ coprime with } p} (x - \operatorname{cis}(2k\pi/p^n) \\ &= \left(\prod_{\text{all } p^{n-1}\text{th roots}} (x-\omega) \right)^{-1} \prod_{\text{all } p^n\text{th roots}} (x-\omega) = \frac{x^{p^n}-1}{x^{p^{n-1}}-1} \\ &= \Phi_p\left(x^{p^{n-1}}\right). \end{aligned}$$