Solution (#164) Let n be a positive integer. We then have

$$x^n - 1 = \prod_{n\text{th roots of unity}} (x - \omega),$$

where the product is taken over all nth roots of unity. Let $\omega = \operatorname{cis}(2k\pi/n)$ be such a root, let h denote the highest common factor of k and n, and let $d = n/h$. Then

$$\omega^r = 1 \iff \operatorname{cis}(2kr\pi/n) = 1 \iff n \text{ divides } kr \iff d \text{ divides } (k/h)r.$$

As d and k/h are coprime (because h is the highest common factor of n and k) then d dividing $(k/h)r$ is equivalent to r being a multiple of d. So the first power of ω which equals 1 is the d or equivalently ω is a primitive dth root. Hence

$$\begin{aligned} x^n - 1 &= \prod_{k=0}^{n-1} (x - \operatorname{cis}(2k\pi/n)) \\ &= \prod_{h|n} \prod_{\substack{k \text{ and } n \text{ have} \\ \text{hcf } h}} (x - \operatorname{cis}(2k\pi/n)) \\ &= \prod_{d|n} \prod_{d\text{th primitive roots}} (x - \omega) \\ &= \prod_{d|n} \Phi_d(x). \end{aligned}$$

We then have that

$$\begin{aligned} x^n + 1 &= \frac{x^{2n} - 1}{x^n - 1} \\ &= \prod_{d|2n} \Phi_d(x) \Big/ \prod_{d|n} \Phi_d(x) \\ &= \prod_{d|2n,\ d\nmid n} \Phi_d(x) \end{aligned}$$