Solution (#171) (i) Let $m \geqslant 2$ be an integer. The remainders r when an integer is divided by m are in the range $0 \leqslant r < m$ – so there are m possible remainders. This means that if the $m+1$ powers

$$1, 2, 4, \ldots, 2^m$$

are divided by m, at least one remainder is repeated.

Suppose that the remainders of 2^k and 2^l (where $0 < k < l$) are the first repeated remainders. We then have that

$$2^k = a_0 m + r_0 \quad \text{and} \quad 2^l = b_0 m + r_0$$

for some integers a_0, b_0, r_0 with $0 \leqslant r_0 < m$. Say that $2r_0 = cm + r_1$ where $0 \leqslant r_1 < m$. Then

$$\begin{aligned} 2^{k+1} &= 2a_0 m + 2r_0 = (2a_0 + c)m + r_1; \\ 2^{l+1} &= 2b_0 m + 2r_0 = (2b_0 + c)m + r_1, \end{aligned}$$

showing that 2^{k+1} and 2^{l+1} also leave the same remainder, namely r_1. In a similar fashion we can see that 2^{k+2} and 2^{l+2} leave the same remainder, and so on, showing that the pattern of remainders now repeats periodically.

(ii) The last digit in a number is the remainder when divided by 10. The powers of 2 begin $2, 4, 8, 16, 32, \ldots$ and so we see that we have a repeating pattern of period 4. This means

$$2^{4k+1} \text{ ends in a 2;} \quad 2^{4k+2} \text{ ends in a 4;} \quad 2^{4k+3} \text{ ends in a 8;} \quad 2^{4k} \text{ ends in a 6.}$$

As $2017 = 4 \times 504 + 1$ then 2^{2017} ends in a 2.