Solution (#173) Let $P(z) = z^2$ and for each possible complex number z_0 consider the sequence

$$z_0,\ P(z_0),\ P(P(z_0)),\ P(P(P(z_0))),\ \ldots$$

formed by repeated application of the function P. Denote by z_n the nth iterate where $n \geqslant 0$. In this case it is easy to check $z_n = (z_0)^{2^n}$.

(i) If $|z_0| > 1$ then $|z_n| = |z_0|^{2^n}$ increases without bound. If $|z_0| \leqslant 1$ then the sequence clearly remains bounded.

(ii) If $|z_0| < 1$ then the sequence z_n converges to 0. There are other values of z_0 for which the sequence converges. For example, in the cases $z_0 = -1$ and $z_0 = i$ the sequence z_n is

$$-1, 1, 1, 1, \ldots, \qquad i, -1, 1, 1, 1, \ldots.$$

More generally, we can see that if z_0 is a 2^kth root of unity, for some $k \geqslant 0$, then the sequence z_n will eventually become constant and equal to 1.

The remaining cases to study are where $z_0 = \operatorname{cis}(\alpha\pi i)$ where α is rational but its denominator is not a power of 2 and where α is irrational. We will treat these cases in (iii) and (iv) below. The generality given in the solution below was not expected from the question; to answer the given question it would be sufficient to simply give one z_0 for each case.

(iii) Suppose that $z_0 = \operatorname{cis}(2^a b\pi i/c)$ where a, b, c are integers and b and c are odd with no common factor. Then

$$z_n = z_0^{2^n} = \operatorname{cis}(2^{n+a} b\pi i/c).$$

From #171, we know that the remainder when 2^n is divided by c eventually becomes periodic, and so z_n is periodic. So the sequence is eventually periodic, but not constant (as c is odd) and so the sequence z_n does not converge.

(iv) Suppose now that $z_0 = \operatorname{cis}(\alpha\pi i)$ where α is irrational. Then $z_n = \operatorname{cis}(2^n\alpha\pi i)$. Note that this sequence never repeats as this would mean for some $m < n$ that

$$\operatorname{cis}(2^m\alpha\pi i) = \operatorname{cis}(2^n\alpha\pi i) \quad \Longrightarrow \quad 2^m\alpha\pi = 2^n\alpha\pi + 2N\pi$$

for some integer N, but then we'd have that α is rational. In particular, z_n is not periodic.

Finally let $\alpha = b_k b_{k-1} \ldots b_1 b_0 . b_{-1} b_{-2} \ldots$ be the binary (base 2) expansion of α. As α is irrational then there are infinitely many 0s and infinitely many 1s amongst the b_r where $r < 0$. This also means that both the strings 01 and 10 appear infinitely often in the expansion and that at least one of the strings 00 and 11 appears infinitely often.

For any n we have, in binary, that

$$2^n\alpha = b_k b_{k-1} \ldots b_{1-n} b_{-n} . b_{-n-1} b_{-n-2} \ldots$$

- If $b_{-n}b_{-n-1} = 10$ then $b_{-n}.b_{-n-1}\ldots$ represents a number in the range 1 to 1.5. In this case $z_n = \operatorname{cis}(2^n\alpha\pi i)$ is in the bottom left quadrant (third quadrant) of the unit circle.
- If $b_{-n}b_{-n-1} = 01$ then $b_{-n}.b_{-n-1}\ldots$ represents a number in the range 0.5 to 1. In this case $z_n = \operatorname{cis}(2^n\alpha\pi i)$ is in the top left quadrant (second quadrant) of the unit circle.

These two happen infinitely often. We also have at least one of the following happening infinitely often.

- If $b_{-n}b_{-n-1} = 00$ then $b_{-n}.b_{-n-1}\ldots$ represents a number in the range 0 to 0.5. In this case $z_n = \operatorname{cis}(2^n\alpha\pi i)$ is in the top right quadrant (first quadrant) of the unit circle.
- If $b_{-n}b_{-n-1} = 11$ then $b_{-n}.b_{-n-1}\ldots$ represents a number in the range 1.5 to 2. In this case $z_n = \operatorname{cis}(2^n\alpha\pi i)$ is in the bottom right quadrant (fourth quadrant) of the unit circle.

That z_n appears infinitely often in three different quadrants is sufficient to ensure that z_n does not converge.

Remark: Whilst this particular choice, $P(z) = z^2$, is a rather simple example of what can happen when iterating with a more general quadratic polynomial, several important features have manifested themselves. There is an interior region from which the iterated sequence remains bounded, an outside region from which said sequence becomes unbounded and a boundary between the two, the so-called **Julia set**. In this particular example the Julia set is just the unit circle, but more generally this set can have incredible complexity. But even on this simple Julia set, we see typical features; at certain points of the set the iterated sequence becomes convergent, at other points it becomes periodic but not convergent, and at other points it is aperiodic and non-convergent. These three types of point appear chaotically on the boundary – that is, given any point of the boundary there are other arbitrarily close boundary points which exhibit each of the three behaviours.