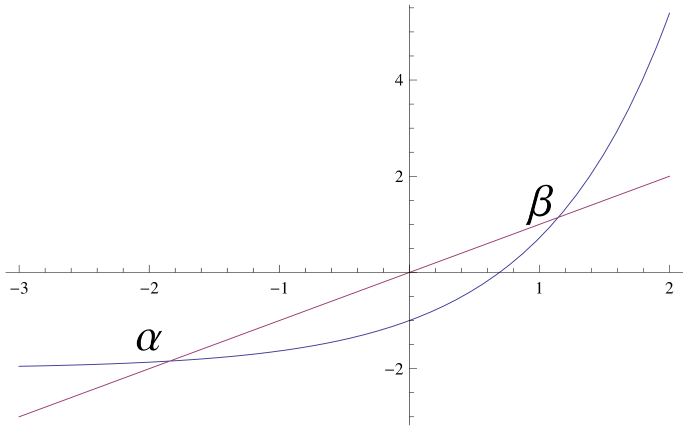
Solution (#178) Let $f(x) = e^x - 2$.



From the sketch above we can see that the equation $e^x - 2 = x$ has two roots, one in the range $-2 < \alpha < -1$ and the other in the range $1 < \beta < 2$. To see this more rigorously we can set $g(x) = e^x - 2 - x$ and note that

$$\begin{aligned} g(-2) &= e^{-2} - 2 + 2 > 0 > e^{-1} - 2 + 1 = g(-1); \\ g(1) &= e - 2 - 1 < 0 < e^2 - 2 - 2 = g(2). \end{aligned}$$

As g is continuous then there is a root in each interval. Further as $g'(x) = e^x - 1$ then g is increasing for $x > 0$ and decreasing for $x < 0$. Hence α and β are the only two roots of g. Also

$$e^{-2} - 1 = g'(-2) < g'(\alpha) < g'(-1) = e^{-1} - 1 \quad \text{showing that } |g'(\alpha)| < 1, \text{ and } \alpha \text{ is attracting,}$$

and

$$e - 1 = g'(1) < g'(\beta) \quad \text{showing that } |g'(\beta)| > 1, \text{ and } \beta \text{ is repelling.}$$

Using cobwebbing sketches, we can see, for the various possible values of x_0, that:

x_n converges to α in an increasing fashion when $x_0 < \alpha$;
$x_n = \alpha$ for all n when $x_0 = \alpha$;
x_n converges to α in a decreasing fashion when $\alpha < x_0 < \beta$;
$x_n = \beta$ for all n when $x_0 = \beta$;
x_n increases in an unbounded fashion when $x_0 > \beta$,

where $x_{n+1} = f(x_n)$.