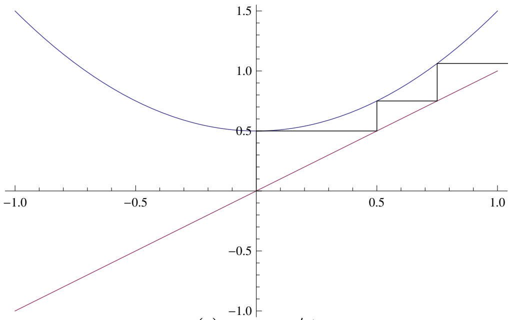
**Solution** (#182) General comments: Let $c$ be a real number. The equation $z^2 + c = z$ has real roots

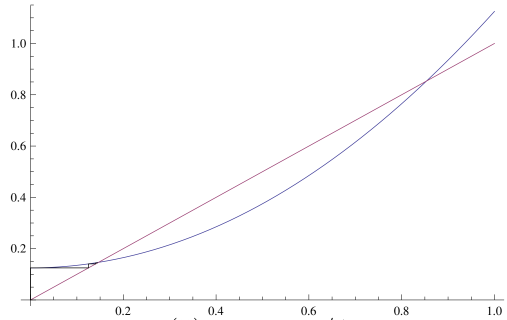
$$\alpha = \left(1 - \sqrt{1-4c}\right)/2, \qquad \beta = \left(1 + \sqrt{1-4c}\right)/2,$$

for $c \leqslant 1/4$ and are distinct when $c < 1/4$. Note that $\beta$ is always repelling and that $\alpha$ is attracting when

$$1 - \sqrt{1-4c} > -1 \iff c > -3/4.$$

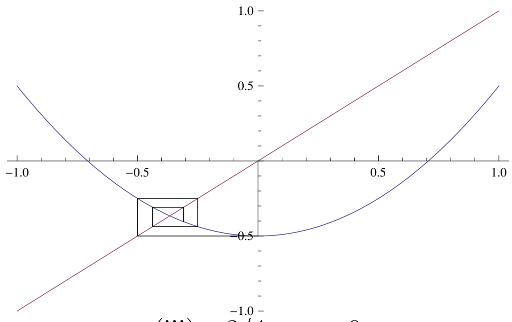



(i) $c > 1/4$

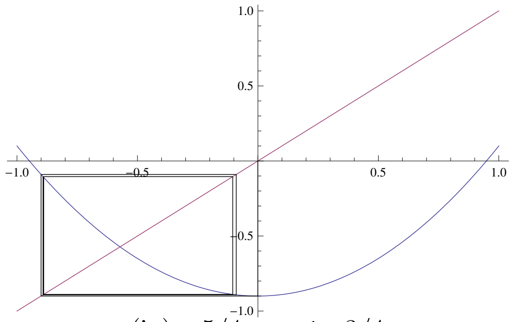



(ii) $0 \leqslant c < 1/4$

(i) For $c > 1/4$ there are no fixed points and $z_n$ increases without bound. (ii) For $0 \leqslant c < 1/4$ then the fixed points $\alpha, \beta$ lie between 0 and 1 with $\alpha$ being attractive. $z_n$ converges in an increasing fashion to $\alpha$.

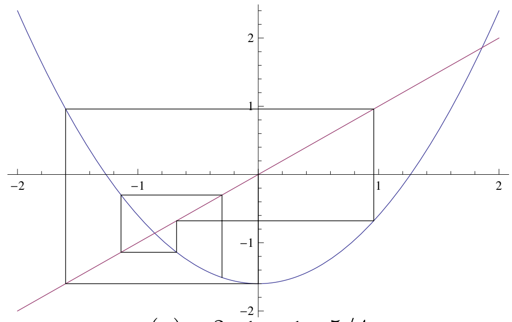



(iii) $-3/4 < c < 0$

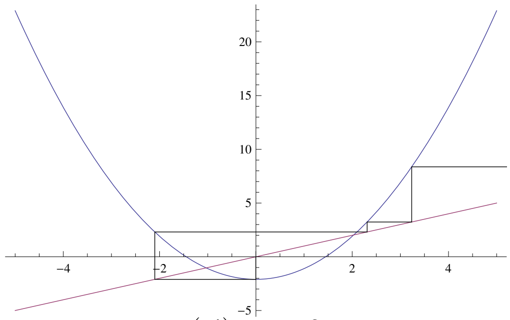



(iv) $-5/4 < c \leqslant -3/4$

(iii) For $-3/4 < c < 0$ then $\alpha$ is negative and attracting, though as $f'(\alpha) = 2\alpha < 0$ we note that $z_n$ converges to $\alpha$ is an oscillatory fashion. (iv) For $-5/4 < c \leqslant -3/4$ then we see $z_n$ oscillating close to two values (an attracting period 2 orbit – see #186). In particular, when $c = -1$, then $(z_n) = (0, -1, 0, -1, 0, -1, \ldots)$.




(v) $-2 \leqslant c \leqslant -5/4$




(vi) $c < -2$

(v) For $-2 \leqslant c \leqslant -5/4$ the behaviour of $z_n$ becomes increasingly erratic. For some values showing chaotic behaviour (e.g. as sketched above) whilst for other values it can be close to periodic. However it does remain the case that $z_n$ remains bounded. We have for each $n$ that $-\beta \leqslant z_n \leqslant \beta$. The reason for this is that the map $f(z) = z^2 + c$ maps this interval back into itself. The largest value $f$ takes on this interval is $f(\beta) = \beta$ and the smallest is $c$. However, $c \geqslant -\beta$ when

$$-c \leqslant \beta \iff -2c \leqslant 1 + \sqrt{1-4c} \iff 4c^2 + 4c + 1 \leqslant 1 - 4c \iff c(c+2) \leqslant 0,$$

which is true for $-2 \leqslant c \leqslant 0$, in which range $c$ is. (vi) For $c < -2$, in the sketch above $z_n$ eventually increases in an unbounded fashion. This is generally the case as seen in #174(iv).

We have exhaustively considered cases covering all possible real numbers, and we see that $z_n(c)$ remains bounded (i.e. $c$ lies in the Mandelbrot set) for $-2 \leqslant c \leqslant 1/4$.