Solution (#187) Say $M = (x, y, z)$ lies on the sphere S. Then the line connecting M and $N = (0, 0, 1)$ can be written parametrically as

$$\mathbf{r}(t) = (0, 0, 1) + t\,(x, y, z - 1)\,.$$

This intersects the plane when $1 + t\,(z - 1) = 0$, i.e. when $t = (1 - z)^{-1}$, i.e. at

$$\mathbf{r}\left(\frac{1}{1-z}\right) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

which is identified with the complex number

$$P = \frac{x + yi}{1 - z}.$$

On the other hand, if $f^{-1}(x, y, z) = x + yi$ then

$$\frac{x + yi}{1 - z} = X + Yi \quad \text{and} \quad x^2 + y^2 + z^2 = 1.$$

Hence $(x - yi) / (1 - z) = X - Yi$ and so

$$X^2 + Y^2 = \left(\frac{x + yi}{1 - z}\right)\left(\frac{x - yi}{1 - z}\right) = \frac{x^2 + y^2}{(1 - z)^2} = \frac{1 - z^2}{(1 - z)^2} = \frac{1 + z}{1 - z} = -1 + \frac{2}{1 - z},$$

giving

$$\frac{2}{1 + X^2 + Y^2} = 1 - z$$

and

$$z = \frac{X^2 + Y^2 - 1}{X^2 + Y^2 + 1}.$$

Then

$$x + yi = \frac{2\,(X + Yi)}{X^2 + Y^2 + 1}$$

and we may compare real and imaginary parts for the result.