Solution (#192) (i) Note that if $c \neq 0$ then

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c^2 z + dc}$$

is a composition of various translations, dilations and inversion – namely (in order)

$$z \mapsto c^2 z, \quad z \mapsto z + dc, \quad z \mapsto z^{-1}, \quad z \mapsto (bc-ad)\, z, \quad z \mapsto z + a/c.$$

If $c = 0$, then $d \neq 0$ and clearly $z \mapsto (a/d)\, z + (b/d)$ is a composition of a dilation and a translation.

(ii) In #119 it was shown that circlines are mapped to circlines by translations, dilations and inversion. It follows that any composition of them would also maps circlines to circlines.

(iii) Consider the maps

$$f(z) = \frac{az+b}{cz+d}, \qquad g(z) = \frac{Az+B}{Cz+D}, \qquad ad - bc \neq 0 \neq AD - BC.$$

Then

$$\begin{aligned} g \circ f(z) &= g(f(z)) \\ &= \frac{A\left(\frac{az+b}{cz+d}\right) + B}{C\left(\frac{az+b}{cz+d}\right) + D} \\ &= \frac{(Aa+Bc)\, z + (Ab+Bd)}{(Ca+Dc)\, z + (Cb+Dd)} \end{aligned}$$

which is another Möbius transformation as

$$(Aa+Bc)(Cb+Dd) - (Ab+Bd)(Ca+Dc) = (ad-bc)(AD-BC) \neq 0. \tag{8.22}$$

Also if

$$\begin{aligned} w &= f(z) = \frac{az+b}{cz+d} \\ czw + dw &= az + b \\ z(cw-a) &= b - dw \\ f^{-1}(w) &= z = \frac{(-d)\, w + b}{cw - a} \end{aligned}$$

and $(-d)(-a) - bc = ad - bc \neq 0$ so that f^{-1} is also a Möbius transformation.