Solution (#198) Let f be a Möbius transformation which maps the unit circle $|z| = 1$ to the real axis and the real axis to the imaginary axis.

The intersections of the unit circle $|z| = 1$ and the real axis are 1 and -1; the intersections of the images (the real and imaginary axis) are 0 and ∞. It follows that (i) $f(1) = 0$ and $f(-1) = \infty$ or (ii) $f(1) = \infty$ and $f(-1) = 0$.

Suppose we are in the first case. It then follows that

$$f(z) = \frac{a(z-1)}{z+1}$$

where a is a non-zero complex number. As i lies on the unit circle then

$$f(i) = \frac{a(i-1)}{i+1} = ai$$

is real. It follows that we can write $a = i\lambda$ where λ is non-zero and real. On the other hand, if

$$f(z) = \frac{i\lambda(z-1)}{z+1}$$

and $|z| = 1$, so that $z\bar{z} = 1$, then

$$\overline{f(z)} = \frac{-i\lambda(\bar{z}-1)}{\bar{z}+1} = \frac{-i\lambda\left(z^{-1}-1\right)}{z^{-1}+1} = \frac{i\lambda(z-1)}{1+z} = f(z)$$

and so $f(z)$ is real, whilst if z is real then

$$\overline{f(z)} = \frac{-i\lambda(\bar{z}-1)}{\bar{z}+1} = \frac{-i\lambda(z-1)}{z+1} = -f(z)$$

and so $f(z)$ is imaginary.

Suppose we are in the second case. It then follows that

$$f(z) = \frac{b(z+1)}{z-1}$$

where b is a non-zero complex number. As i lies on the unit circle then

$$f(i) = \frac{b(i+1)}{i-1} = -bi$$

is real. It follows that we can write $b = i\mu$ where μ is non-zero and real. On the other hand, if

$$f(z) = \frac{i\mu(z+1)}{z-1}$$

and $|z| = 1$, so that $z\bar{z} = 1$, then

$$\overline{f(z)} = \frac{-i\mu(\bar{z}+1)}{\bar{z}-1} = \frac{-i\mu\left(z^{-1}+1\right)}{z^{-1}-1} = \frac{i\mu(z+1)}{z-1} = f(z)$$

and so $f(z)$ is real, whilst if z is real then

$$\overline{f(z)} = \frac{-i\mu(\bar{z}+1)}{\bar{z}-1} = \frac{-i\mu(1+z)}{1-z} = -f(z)$$

and so $f(z)$ is imaginary.