Solution (#207) Let $f(z) = (az+b)/(cz+d)$ be a Möbius transformation, not equal to the identity.

Note that f fixes ∞ if and only if $c = 0$ and that any finite fixed points z satisfy

$$cz^2 + (d-a)z - b = 0. \tag{8.25}$$

If $c = 0$ then ∞ is fixed and (8.25) can at most have one further solution.

If $c \neq 0$ then ∞ is not fixed and (8.25) can have one or two solutions.

(i) If ∞ is fixed then $c = 0$. We see $b/(d-a)$ is a further fixed point unless $a = d$. Hence the map f in this case is $z \mapsto z + b/d$ which is a translation.

(ii) Again if ∞ is fixed then $c = 0$. If 0 is also fixed then $b = 0$. Hence the map f in this case is $z \mapsto (a/c)z$ which is a dilation.

(iii) Suppose now that f fixes precisely one point, call it α. Let g be a Möbius transformation such that $g(\infty) = \alpha$. Then $g^{-1}fg$ is also a Möbius transformation by #192 (iii); further

$$\begin{aligned} & g^{-1}fg(z) = z \\ \iff & f(g(z)) = g(z) \\ \iff & g(z) = \alpha \\ \iff & z = \infty. \end{aligned}$$

So $g^{-1}fg$ is a Möbius transformation fixing only ∞ and so is a translation.

(iv) Suppose now that f fixes two points, α and β. Let g be a Möbius transformation such that $g(\infty) = \alpha$ and $g(\beta) = 0$. Then $g^{-1}fg$ is also a Möbius transformation by #192 (iii); further

$$\begin{aligned} & g^{-1}fg(z) = z \\ \iff & f(g(z)) = g(z) \\ \iff & g(z) = \alpha \text{ or } \beta \\ \iff & z = \infty \text{ or } 0. \end{aligned}$$

So $g^{-1}fg$ is a Möbius transformation fixing ∞ and 0 and so is a dilation.