Solution (#213) Let

$$f(z) = \frac{\operatorname{cis}\theta\ (z-a)}{1-\bar{a}z} \quad \text{where } \theta \in \mathbb{R} \text{ and } |a| < 1.$$

Then this is a Möbius transformation as

$$(\operatorname{cis}\theta) \times 1 - (-a\operatorname{cis}\theta)(-\bar{a}) = \operatorname{cis}\theta\left(1 - |a|^2\right) \neq 0$$

and clearly $f(a) = 0$. Further if $|z| = 1$ then

$$|f(z)| = \left|\frac{\operatorname{cis}\theta\ (z-a)}{1-\bar{a}z}\right| = \frac{|z-a|\,|\bar{z}|}{|1-\bar{a}z|} = \frac{|1-a\bar{z}|}{|1-\bar{a}z|} = \left|\frac{1-a\bar{z}}{1-\bar{a}z}\right| = 1$$

as the numerator is the conjugate of the denominator.