Solution (#214) Set

$$f(z) = w = \frac{\operatorname{cis}\theta\ (z-a)}{1-\bar{a}z}.$$

Then, solving for z we have

$$(1-\bar{a}z)\, w\operatorname{cis}(-\theta) = z - a$$
$$\implies w\operatorname{cis}(-\theta) + a = z\left(1 + \bar{a}w\operatorname{cis}(-\theta)\right)$$

and hence

$$z = f^{-1}(w) = \frac{w\operatorname{cis}(-\theta) + a}{1 + \bar{a}w\operatorname{cis}(-\theta)} = \frac{\operatorname{cis}(-\theta)\left(w + a\operatorname{cis}\theta\right)}{1 - w\left(\overline{-a\operatorname{cis}\theta}\right)}$$

which is of the same form as f but with $-\theta$ replacing θ and $-a\operatorname{cis}\theta$ replacing a. Likewise

$$\begin{aligned} g(f(z)) &= \frac{\operatorname{cis}\alpha\left(\frac{\operatorname{cis}\theta\,(z-a)}{1-\bar{a}z} - b\right)}{1 - \bar{b}\left(\frac{\operatorname{cis}\theta\,(z-a)}{1-\bar{a}z}\right)} \\ &= \frac{\operatorname{cis}\alpha\left(\operatorname{cis}\theta\ (z-a) - b\,(1-\bar{a}z)\right)}{1 - \bar{a}z - \bar{b}\operatorname{cis}\theta\ (z-a)} \\ &= \frac{\operatorname{cis}\alpha\left(\left(\operatorname{cis}\theta + \bar{a}b\right)z - \left(b + a\operatorname{cis}\theta\right)\right)}{\left(1 + a\bar{b}\operatorname{cis}\theta\right) - \left(\bar{a} + \bar{b}\operatorname{cis}\theta\right)\ z} \\ &= \frac{\operatorname{cis}\alpha\ (z-A)}{1-\bar{A}z} \end{aligned}$$

where

$$A = \frac{b + a\operatorname{cis}\theta}{\operatorname{cis}\theta + \bar{a}b} \quad \text{and} \quad \bar{A} = \frac{\bar{b} + \bar{a}\operatorname{cis}\ (-\theta)}{\operatorname{cis}\ (-\theta) + a\bar{b}} = \frac{\bar{a} + \bar{b}\operatorname{cis}\theta}{1 + a\bar{b}\operatorname{cis}\theta}.$$

Further note that

$$A = \frac{b + a\operatorname{cis}\theta}{\operatorname{cis}\theta + \bar{a}b} = \frac{b\operatorname{cis}\ (-\theta) - (-a)}{1 - (-\bar{a})\, b\operatorname{cis}\ (-\theta)} = h\left(b\operatorname{cis}\ (-\theta)\right)$$

where

$$h = \frac{z - (-a)}{1 - (-\bar{a})\, z}$$

is a map, which like f maps the unit disc into the unit disc and $|b\operatorname{cis}\ (-\theta)| = |b| < 1$; in particular then $|A| < 1$.