Solution (#218) Our Möbius transformation f has the form

$$f(z) = \frac{\operatorname{cis}\theta\ (z-a)}{1-\bar{a}z}$$

Note that for p, q in the hyperbolic plane

$$\left|\frac{f(q)-f(p)}{1-\overline{f(p)}f(q)}\right| = \left|\frac{\frac{\operatorname{cis}\theta\,(q-a)}{1-\bar{a}q} - \frac{\operatorname{cis}\theta\,(p-a)}{1-\bar{a}p}}{1-\left(\frac{\operatorname{cis}(-\theta)\,(\bar{p}-\bar{a})}{1-a\bar{p}}\right)\left(\frac{\operatorname{cis}\theta\,(q-a)}{1-\bar{a}q}\right)}\right|.$$

If we multiply numerator and denominator by $|1-\bar{a}q|\,|1-\bar{a}p| = |1-\bar{a}q|\,|1-a\bar{p}|$ and recall $|\operatorname{cis}\theta| = 1$, $\operatorname{cis}(-\theta) = (\operatorname{cis}\theta)^{-1}$, we find the above simplifies to

$$\left|\frac{(q-a)(1-\bar{a}p) - (p-a)(1-\bar{a}q)}{(1-a\bar{p})(1-\bar{a}q) - (\bar{p}-\bar{a})(q-a)}\right|.$$

The numerator now simplifies as

$$q + a\bar{a}p - p - a\bar{a}q = (p-q)(a\bar{a}-1)$$

and the denominator simplifies as

$$1 + a\bar{a}\bar{p}q - \bar{p}q - a\bar{a} = (\bar{p}q-1)(a\bar{a}-1).$$

Hence, noting $|a|^2 = a\bar{a} < 1$

$$\begin{aligned} d_D(f(p), f(q)) &= 2\tanh^{-1}\left|\frac{f(q)-f(p)}{1-\overline{f(p)}f(q)}\right| \\ &= 2\tanh^{-1}\left|\frac{(p-q)(a\bar{a}-1)}{(\bar{p}q-1)(a\bar{a}-1)}\right| \\ &= 2\tanh^{-1}\left|\frac{p-q}{1-\bar{p}q}\right| \\ &= d_D(p,q) \end{aligned}$$

as required.