Solution (#220) As we are to measure distances in H in such a way as to make g an isometry, and likewise g^{-1} an isometry, then the lines in D will map under g^{-1} to the lines in H. Now

$$g(z) = \frac{z-i}{z+i}$$

and

$$g^{-1}(z) = \frac{i(z+1)}{1-z}.$$

As g^{-1} is a Möbius transformation then it is conformal and maps circlines to circlines. Note also that g^{-1} maps the unit circle (the boundary of D) to the real axis (the boundary of H). Now the lines in D are the (arcs of) circlines which intersect the unit circle at right angles. So the lines in H are the (arcs of) circlines which intersect the real axis – i.e. they are either half-lines perpendicular to the real axis or semi-circles which cut the real axis in right angles.

(ii) If the map g from D to H is to be an isometry then, for p, q in H, we have

$$\begin{aligned} d_H(p,q) &= d_D\left(g^{-1}(p), g^{-1}(q)\right) \\ &= d_D\left(\frac{i(p+1)}{1-p}, \frac{i(q+1)}{1-q}\right) \\ &= 2\tanh^{-1}\left|\frac{\frac{i(p+1)}{1-p} - \frac{i(q+1)}{1-q}}{1 - \frac{(p+1)}{1-p}\frac{(\bar{q}+1)}{1-\bar{q}}}\right| \\ &= 2\tanh^{-1}\left|\frac{(p+1)(1-q) - (q+1)(1-p)}{(1-p)(1-\bar{q}) - (p+1)(\bar{q}+1)}\right| \end{aligned}$$

as $|1-q| = |1-\bar{q}|$. Then, simplifying,

$$d_H(p,q) = 2\tanh^{-1}\left|\frac{2(p-q)}{2(p-\bar{q})}\right| = 2\tanh^{-1}\left|\frac{p-q}{p-\bar{q}}\right|.$$

As

$$\tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

then we also have

$$d_H(p,q) = \ln\left|\frac{1+\left(\frac{p-q}{p-\bar{q}}\right)}{1-\left(\frac{p-q}{p-\bar{q}}\right)}\right| = \ln\left\{\frac{|p-q| + |p-\bar{q}|}{|p-q| - |p-\bar{q}|}\right\}$$