Solution (#223) Taking the hyperbolic triangle described in #221, we have from the hyperbolic sine rule that

$$\rho = \sinh b \sinh c \sin \hat{A} = \sinh a \sinh b \sin \hat{C} = \sinh a \sinh c \sin \hat{B}.$$

To avoid notational messes we shall again write $\alpha = \cosh a$, $\beta = \cosh b$, $\gamma = \cosh c$. From the cosine rule for $\cos \hat{A}$ we have

$$\cos \hat{A} = \frac{\beta\gamma - \alpha}{\sinh b \sinh c} = \frac{\sin \hat{A}}{\rho} \left(\beta\gamma - \alpha\right).$$

Making similar use of the rules for $\cos \hat{A}, \cos \hat{B}, \cos \hat{C}$, we have that

$$\begin{aligned} \frac{\cos \hat{B} \cos \hat{C} + \cos \hat{A}}{\sin \hat{B} \sin \hat{C}} &= \frac{1}{\rho^2} \left(\alpha\gamma - \beta\right)\left(\alpha\beta - \gamma\right) + \frac{\sin \hat{A}}{\rho \sin \hat{B} \sin \hat{C}} \left(\beta\gamma - \alpha\right) \\ &= \frac{1}{\rho^2} \left(\alpha\gamma - \beta\right)\left(\alpha\beta - \gamma\right) + \frac{\sinh^2 a}{\rho^2} \left(\beta\gamma - \alpha\right) \\ &= \frac{\left(\alpha\gamma - \beta\right)\left(\alpha\beta - \gamma\right) + \left(\alpha^2 - 1\right)\left(\beta\gamma - \alpha\right)}{\rho^2} \\ &= \frac{\alpha(2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2 + 1)}{\rho^2}. \end{aligned}$$

However we showed in the solution of # 222 that $\rho^2 = \sinh^2 b \sinh^2 c \sin^2 \hat{A} = 2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2 + 1$. Thus

$$\frac{\cos \hat{B} \cos \hat{C} + \cos \hat{A}}{\sin \hat{B} \sin \hat{C}} = \alpha = \cosh a$$

as required. As $\cosh a > 1$ it follows that

$$\cos \hat{A} > \sin \hat{B} \sin \hat{C} - \cos \hat{B} \cos \hat{C} = -\cos(\hat{B} + \hat{C}) = \cos\left(\pi - \hat{B} - \hat{C}\right).$$

As $\cos \theta$ is decreasing for $0 \leqslant \theta \leqslant \pi$ we have that

$$\hat{A} < \pi - \hat{B} - \hat{C}$$

and we have $\hat{A} + \hat{B} + \hat{C} < \pi$.