Solution (#1587) Define

$$f(x,y)=\sqrt{y} \qquad \text{for real } x \text{ and } y\geqslant 1.$$

Then

$$\begin{aligned} |f(x,y_1)-f(x,y_2)| &= |\sqrt{y_1}-\sqrt{y_2}| \\ &= \frac{|y_1-y_2|}{|\sqrt{y_1}+\sqrt{y_2}|} \\ &\leqslant \frac{1}{2}|y_1-y_2| \qquad \text{when } y\geqslant 1. \end{aligned}$$

Consider the case when

$$f(x,y)=\sqrt{y} \qquad \text{for real } x \text{ and } y>0.$$

Fix $y_1=\varepsilon$ and $y_2=\varepsilon/2$ where $\varepsilon>0$. Then

$$\frac{|f(x,y_1)-f(x,y_2)|}{|y_1-y_2|}=\frac{\left|\sqrt{\varepsilon}-\sqrt{\varepsilon/2}\right|}{|\varepsilon-\varepsilon/2|}=\frac{2-\sqrt{2}}{\sqrt{\varepsilon}}.$$

As ε becomes small, the expression on the RHS increases without bound. So there is no real number K greater than all such expressions.