Solution (#1597) If $x(t)$ is the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1+x)\, e^{x}, \quad x(0) = 0,$$

then for any time t_0 and corresponding $x_0 = x(t_0)$ we have

$$t_0 = \int_0^{t_0} \mathrm{d}t = \int_0^{x_0} \frac{\mathrm{d}x}{(1+x)\, e^{x}}.$$

If $x(T) = 1$ then

$$T = \int_0^1 \frac{\mathrm{d}x}{(1+x)\, e^{x}}.$$

In the range $0 < x < 1$ then $\frac{1}{2} < (1+x)^{-1} < 1$ and so

$$\begin{aligned} T &= \int_0^1 \frac{\mathrm{d}x}{(1+x)\, e^{x}} > \int_0^1 \frac{\mathrm{d}x}{2e^{x}} = \left[-\frac{e^{-x}}{2}\right]_0^1 = \frac{1}{2} - \frac{1}{2e}; \\ T &= \int_0^1 \frac{\mathrm{d}x}{(1+x)\, e^{x}} < \int_0^1 \frac{\mathrm{d}x}{e^{x}} = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}, \end{aligned}$$

as required.