Solution (#1609) A particle P moves in the xy-plane. Its co-ordinates $x(t)$ and $y(t)$ satisfy the equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y \qquad \text{and} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = x - y,$$

and at time $t = 0$ the particle is at $(1, 0)$. By the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x+y}{x-y}.$$

If we set $y(x) = xv(x)$ so that $y' = v + xv'$ then we see

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+v}{1-v}$$

which rearranges to

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+v^2}{1-v}$$

which is a separable differential equation. Then

$$\int \left(\frac{1-v}{1+v^2}\right) \mathrm{d}v = \int \frac{\mathrm{d}x}{x}$$

and integrating we see

$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) + \text{const.} = \ln x$$

Initially $x = 1$, $y = 0$ and $v = 0$ so that the above constant equals 0. The particle therefore travels on the curve

$$\begin{aligned} \tan^{-1}\left(\frac{y}{x}\right) &= \ln x + \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) \\ &= \frac{1}{2}\ln(x^2+y^2). \end{aligned}$$

Now if we change to polar co-ordinates ($x = r\cos\theta$, $y = r\sin\theta$) our curve's equation becomes

$$\theta = \frac{1}{2}\ln r^2 = \ln r \qquad \Longrightarrow \qquad r = e^\theta.$$

Note that the equation $y'(t) = x(t) + y(t)$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^\theta \sin\theta) = e^\theta(\cos\theta + \sin\theta)$$

and via the chain rule we then have

$$e^\theta(\cos\theta + \sin\theta)\theta'(t) = e^\theta(\cos\theta + \sin\theta)$$

so that $\theta'(t) = 1$. At $\theta = 0$ when $t = 0$ then $\theta = t$ and we have

$$(x(t), y(t)) = (e^t\cos t, e^t \sin t).$$

A sketch of the curve for $0 \leqslant t \leqslant 2\pi$ is given below.