Solution (#1616) Let α and β be distinct real numbers. Say that y satisfies

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-(\alpha+\beta)\frac{\mathrm{d}y}{\mathrm{d}x}+\alpha\beta y=0.$$

Let

$$f(x)=y'(x)-\alpha y(x) \qquad \text{and} \qquad g(x)=y'(x)-\beta y(x).$$

Then

$$\begin{aligned} f'(x)-\beta f(x) &= (y''(x)-\alpha y'(x))-\beta\,(y'(x)-\alpha y(x)) \\ &= y''(x)-(\alpha+\beta)y'(x)+\alpha\beta y(x)=0 \end{aligned}$$

and

$$\begin{aligned} g'(x)-\alpha g(x) &= (y''(x)-\beta y'(x))-\alpha\,(y'(x)-\beta y(x)) \\ &= y''(x)-(\alpha+\beta)y'(x)+\alpha\beta y(x)=0. \end{aligned}$$

So

$$\begin{aligned} f(x) &= y'(x)-\alpha y(x)=Ae^{\beta x}, \\ g(x) &= y'(x)-\beta y(x)=Be^{\alpha x}, \end{aligned}$$

for some constants A, B. Subtracting these equations we find

$$y(x)=\frac{A}{\beta-\alpha}e^{\beta x}-\frac{B}{\beta-\alpha}e^{\alpha x}.$$