Solution (#1653) Note that the general solution of $y'' + y = 0$ is $y = A\cos x + B\sin x$.

- $y'' + y = \sin^2 x$. If we write
$$\sin^2 x = \frac{1}{2}(1 - \cos 2x),$$
then it seems reasonable to seek a particular solution of the form
$$y = A\cos 2x + B\sin 2x + C.$$
Substituting this y into the given DE we obtain
$$(A - 4A)\cos 2x + (B - 4B)\sin 2x + C = \frac{1}{2} - \frac{1}{2}\cos 2x.$$
Comparing coefficients we see that $A = 1/6$, $B = 0$ and $C = 1/2$. Hence a particular solution is
$$y = \frac{1}{6}\cos 2x + \frac{1}{2}.$$
Note that we could equally have sought a particular solution of the form
$$y = A\cos^2 x + B\sin x\cos x + C\sin^2 x.$$

- $y'' + y = \sin x$. Note that $\sin x$ is part of the complementary function. Consequently we will seek a particular solution of the form
$$y = Ax\cos x + Bx\sin x.$$
We then have
$$y' = A(\cos x - x\sin x) + B(\sin x + x\cos x); \qquad y'' = A(-2\sin x - x\cos x) + B(2\cos x - x\sin x).$$
Substituting this y into the given DE we obtain
$$A(-2\sin x - x\cos x) + B(2\cos x - x\sin x) + (Ax\cos x + Bx\sin x) = \sin x,$$
which simplifies to
$$-2A\sin x + 2B\cos x = \sin x.$$
Comparing coefficients we have that $A = -1/2$ and $B = 0$ so that a particular solution is
$$y = -\frac{x}{2}\sin x.$$

- $y'' + y = x\sin 2x$. We will seek a particular solution of the form
$$y = Ax\sin 2x + Bx\cos 2x + C\sin 2x + D\cos 2x.$$
We then have
$$\begin{aligned} y' &= A(\sin 2x + 2x\cos 2x) + B(\cos 2x - 2x\sin 2x) + 2C\cos 2x - 2D\sin 2x; \\ y'' &= A(4\cos 2x - 4x\sin 2x) + B(-4\sin 2x - 4x\cos 2x) - 4C\sin 2x - 4D\cos 2x. \end{aligned}$$
Substituting this y into the given DE and simplifying we obtain
$$(A - 4A)\,x\sin 2x + (B - 4B)\,x\cos 2x + (C - 4B - 4C)\sin 2x + (4A + D - 4D)\cos 2x = x\sin 2x.$$
Comparing coefficients we have
$$-3A = 1; \qquad -3B = 0 \qquad 3C - 4B = 0; \qquad 4A - 3D = 0.$$
Hence $B = C = 0$ and $A = -1/3$, $D = -4/9$, and a particular solution is
$$y = -\frac{1}{3}x\sin 2x - \frac{4}{9}\cos 2x.$$