Solution (#1659) Consider the following boundary-value problems.

(i) $y'' = \pi^2 y, \quad y(0) = 1,\ y(1) = -1$. The DE has general solution

$$y = Ae^{\pi x} + Be^{-\pi x}.$$

The boundary conditions then require

$$A + B = 1, \qquad Ae^{\pi} + Be^{-\pi} = -1$$

and so

$$A = -\left(\frac{e^{-\pi}+1}{e^{\pi}-e^{-\pi}}\right), \qquad B = \frac{e^{\pi}+1}{e^{\pi}-e^{-\pi}}.$$

In particular we have a unique solution.

(ii) $y'' = -\pi^2 y, \quad y(0) = 1,\ y(1) = -1$. The DE has general solution

$$y = A\sin\pi x + B\cos\pi x.$$

The boundary conditions then require

$$B = 1, \qquad -B = -1.$$

Thus there are infinitely many solutions

$$y = A\sin\pi x + \cos\pi x.$$

(iii) $y'' = -\pi^2 y, \quad y(0) = 1,\ y(1) = 0$. The DE has general solution

$$y = A\sin\pi x + B\cos\pi x.$$

The boundary conditions then require

$$B = 1, \qquad -B = 0.$$

As these requirements are contradictory there is no solution to this boundary-value problem.