Solution (#1662) Let $z = \ln x$. Then $\mathrm{d}z/\mathrm{d}x = 1/x$ and $\mathrm{d}x/\mathrm{d}z = x$, and by the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}z}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}z}\right) = \frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \frac{1}{x}\left(\frac{-1}{x^2}\frac{\mathrm{d}x}{\mathrm{d}z}\right)\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z}.$$

So

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}; \qquad x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}.$$

Hence our DE – as a DE now involving y and z – has become

$$0 = \left(\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right) + \frac{\mathrm{d}y}{\mathrm{d}z} + y = \frac{\mathrm{d}^2y}{\mathrm{d}z^2} + y.$$

The general solution of this DE is

$$\begin{aligned} y &= A\sin z + B\cos z \\ &= A\sin(\ln x) + B\cos(\ln x). \end{aligned}$$