Solution (#1666) Consider the simultaneous differential equations

$$f'(x) - g'(x) - 2f(x) = e^{-2x}; \qquad 3f'(x) - 2g'(x) - g(x) = 0; \qquad f(0) = g(0) = 0.$$

From the first equation

$$f'(x) - 2f(x) - e^{-2x} = g'(x)$$

and so

$$3f'(x) - 2(f'(x) - 2f(x) - e^{-2x}) - g(x) = 0$$

which rearranges to

$$g(x) = f'(x) + 4f(x) + 2e^{-2x}. \tag{13.5}$$

Differentiating we get

$$g'(x) = f''(x) + 4f'(x) - 4e^{-2x}$$

and so

$$f'(x) - (f''(x) + 4f'(x) - 4e^{-2x}) - 2f(x) = e^{-2x}$$

which rearranges to

$$3e^{-2x} = f''(x) + 3f'(x) + 2f(x).$$

The complementary function has the form $f(x) = Ae^{-x} + Be^{-2x}$ and so we try a particular solution of the form $f(x) = Cxe^{-2x}$. Then

$$3e^{-2x} = C(4x-4)e^{-2x} + 3C(-2x+1)e^{-2x} + 2Cxe^{-2x} = -Cxe^{-2x}$$

and we see $C = -3$. So

$$f(x) = Ae^{-x} + Be^{-2x} - 3xe^{-2x}.$$

But we also know $f(0) = 0$ and from (13.5) $f'(0) = g(0) - 4f(0) - 2 = -2$. So

$$A + B = 0 \qquad \text{and} \qquad -A - 2B - 3 = -2$$

giving $A = 1$ and $B = -1$. So

$$f(x) = e^{-x} - e^{-2x} - 3xe^{-2x}$$

and by (13.5)

$$\begin{aligned} g(x) &= (-e^{-x} + 2e^{-2x} - 3e^{-2x} + 6xe^{-2x}) + 4(e^{-x} - e^{-2x} - 3xe^{-2x}) + 2e^{-2x} \\ &= 3e^{-x} - 3e^{-2x} - 6xe^{-2x}. \end{aligned}$$