Solution (#1669) Let $a_0, a_1, \ldots, a_k$ be real constants and suppose that the polynomial

$$z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0 = 0$$

has k distinct real roots $\gamma_1, \gamma_2, \ldots, \gamma_k$.

Then the general solution of the DE

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}\frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0,$$

is

$$y = A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x} + \cdots + A_k e^{\gamma_k x}.$$

Let $c_0, c_1, c_2, \ldots, c_{k-1}$ be real numbers. To meet the initial conditions

$$y(0) = c_0, \quad y'(0) = c_1, \quad \ldots, \quad y^{(k-1)}(0) = c_{k-1},$$

we need

$$A_1 + A_2 + \cdots + A_k = c_0;$$
$$A_1\gamma_1 + A_2\gamma_2 + \cdots + A_k\gamma_k = c_1;$$
$$A_1(\gamma_1)^2 + A_2(\gamma_2)^2 + \cdots + A_k(\gamma_k)^2 = c_2;$$

up to

$$A_1(\gamma_1)^{k-1} + A_2(\gamma_2)^{k-1} + \cdots + A_k(\gamma_k)^{k-1} = c_{k-1}.$$

We can rewrite these k equations as the single matrix equation

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \vdots & \vdots & \ddots & \vdots \\ (\gamma_1)^{k-1} & (\gamma_2)^{k-1} & \cdots & (\gamma_k)^{k-1} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}.$$

By Example 3.165 the Vandermonde matrix on the right is invertible and so we can uniquely solve the system to find $A_1, \ldots, A_k$.