Solution (#1674) Let $x_1(t), \ldots, x_n(t)$ be n variables satisfying the DEs

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}x_1 + \cdots + a_{1n}x_n; \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = a_{12}x_1 + \cdots + a_{2n}x_n; \qquad \cdots \qquad \frac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}x_1 + \cdots + a_{nn}x_n.$$

These can be more succinctly written as a single differential equation

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = A\mathbf{r}$$

in a vector $\mathbf{r} = (x_1, x_2, \ldots, x_n)^T$ where $A = (a_{ij})$.

Say that A is diagonalizable with eigenbasis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If we set

$$P = (\mathbf{v}_1 \,|\, \mathbf{v}_2 \,|\, \ldots \,|\, \mathbf{v}_n)$$

then

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

If we set $\mathbf{R} = P^{-1}\mathbf{r} = (X_1, X_2, \ldots, X_n)$ then we have

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}P^{-1}\mathbf{r} = P^{-1}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right) = P^{-1}A\mathbf{r} = P^{-1}AP\mathbf{R}.$$

This matrix equation reads as

$$\frac{\mathrm{d}X_i}{\mathrm{d}t} = \lambda_i X_i \qquad i = 1, 2, \ldots, n,$$

and so $X_i = A_i e^{\lambda_i t}$ where $A_i = X_i(0)$. So

$$\mathbf{R}(t) = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} A_1 e^{\lambda_1 t} \\ \vdots \\ A_n e^{\lambda_n t} \end{pmatrix} = A_1 e^{\lambda_1 t}\mathbf{e}_1^T + \cdots + A_n e^{\lambda_n t}\mathbf{e}_n^T.$$

Finally

$$\begin{aligned} \mathbf{r}(t) &= P\mathbf{R}(t) \\ &= A_1 e^{\lambda_1 t} P\mathbf{e}_1^T + \cdots + A_n e^{\lambda_n t} P\mathbf{e}_n^T \\ &= A_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + A_n e^{\lambda_n t}\mathbf{v}_n. \end{aligned}$$