Solution (#1687) (i) Consider the system

$$x' = y, \qquad y' = 1 - x^4 - y^2.$$

For equilibrium we need $x' = y' = 0$ and so $y = 0$ and $x = \pm 1$.

(ii) If we set $x = 1 + u$ and $y = v$ then the system becomes $u' = v$ and

$$v' = 1 - (u-1)^4 - v^2 = 4u + (-6u^2 + 4u^3 - u^4 + v^2).$$

Now if we assume u and v remain suitably small that we can ignore second order terms, then the system linearizes to

$$u' = v, \qquad v' = 4u.$$

Then $u'' = 4u$ which has solution

$$u(t) = A\cos 2t + B\sin 2t = \alpha\cos(2t + \varepsilon)$$

for some $A, B, \alpha, \varepsilon$ and we have

$$v(t) = u'(t) = 2\alpha\sin(2t + \varepsilon).$$

Hence we see that $4u^2 + v^2 = 4\alpha^2$ is constant or, returning to our original co-ordinates, that

$$4(x-1)^2 + y^2 = \text{const}.$$

which are ellipses centred on $(1, 0)$.