Solution (#1688) The only equilibrium is therefore $(0,0)$. If we linearize the linear system about $(0,0)$ then we have

$$x' = x + 2y, \qquad y' = -2x - y$$

We find

$$x'' = -3x, \qquad y'' = -3y$$

and thus both x and y have trigonometric solutions which we remain close to the origin for any small perturbation. In particular $(0,0)$ is a stable equilibrium.

Finally we note

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\left[x^2 + xy + y^2 + \frac{1}{8}x^4\right] \\ = \quad & 2xx' + xy' + x'y + 2yy' + \frac{1}{2}x^3x' \\ = \quad & \left(2x + y + \frac{1}{2}x^3\right)(x+2y) + (x+2y)(-2x - y - \frac{1}{2}x^3) = 0 \end{aligned}$$

and so

$$x^2 + xy + y^2 + \frac{1}{8}x^4$$

is constant as the system moves. Graphs of these curves near the origin approximate to graphs of the form

$$x^2 + xy + y^2 = \text{const.}.$$

These curves are ellipses as seen in #1130. A sketch of a family of such curves is given below.