Solution (#1715) In Example 6.71 we found that

$$f(x) = x^2 e^{-x}$$

is a solution to

$$x^2 f''(x) = \left(x^2 - 4x + 2\right) f(x).$$

Using the methods of §6.4 we set

$$f(x) = u(x) x^2 e^{-x},$$

so that

$$\dot{f}'(x) = u'(x)x^2e^{-x} + u(x)(2x - x^2)e^{-x}, \qquad f''(x) = u''(x)x^2e^{-x} + 2u'(x)(2x - x^2)e^{-x} + u(x)\left(x^2 - 4x + 2\right)e^{-x}.$$

So we have

$$x^2\left[u''(x)x^2e^{-x} + 2u'(x)(2x - x^2)e^{-x} + u(x)\left(x^2 - 4x + 2\right)e^{-x}\right] = \left(x^2 - 4x + 2\right)u(x)x^2e^{-x}$$

which simplifies to

$$u''(x)x^2 + 2u'(x)(2x - x^2) = 0.$$

This is a separable DE which rearranges to

$$\frac{u''(x)}{u'(x)} = \frac{2(x^2 - 2x)}{x^2} = 2 - \frac{4}{x}.$$

Integrating we find

$$\ln u'(x) = 2x - 4\ln x + \text{const.}.$$

So

$$u'(x) = Ax^{-4}e^{2x}$$

for some constant A. We then have an independent solution

$$f(x) = x^2 e^{-x} \int_1^x t^{-4} e^{2t} \mathrm{d}t.$$

For $0 < \varepsilon < 1$ we note

$$\left|\int_1^\varepsilon t^{-4}e^{2t}\mathrm{d}t\right| = \left|\int_\varepsilon^1 t^{-4}e^{2t}\mathrm{d}t\right| \geqslant \left|\int_\varepsilon^1 t^{-4}\mathrm{d}t\right| = \left[\frac{t^{-3}}{-3}\right]_\varepsilon^1 = \frac{\varepsilon^{-3} - 1}{3}$$

and so for $0 < \varepsilon < 1$ we also have

$$|f(\varepsilon)| \geqslant \varepsilon^2 e^{-\varepsilon}\left(\frac{\varepsilon^{-3} - 1}{3}\right) \geqslant \left(\frac{\varepsilon^{-1} - \varepsilon^2}{3e}\right).$$

We know that x^{-1} does not have a Laplace transform, as $x^{-1}e^{-sx}$ is not integrable on $(0,1)$ for any s, and similarly this solution f will not have a convergent Laplace transform either.