Solution (#1718) The DE can be rewritten as

$$f'(x) - 2(f * g)(x) = e^{2x}, \qquad f(0) = 0,$$

where $g(x) = e^{-x}$. Applying the Laplace transform to this equation leads to

$$s\bar{f}(s) - \frac{2\bar{f}(s)}{s+1} = \frac{1}{s-2}.$$

Rearranging gives

$$\bar{f}(s) = \frac{s+1}{(s-2)\,(s^2+s-2)} = \frac{s+1}{(s-2)(s+2)\,(s-1)}.$$

Using partial fractions we have

$$\bar{f}(s) = \frac{3/4}{s-2} - \frac{1/12}{s+2} - \frac{2/3}{s-1}.$$

Hence

$$f(x) = \frac{3}{4}e^{2x} - \frac{1}{12}e^{-2x} - \frac{2}{3}e^{x}.$$