

Solution (#1721) Let $0 < a < 1$. Consider the boundary-value problem $f''(x) = \delta(x - a)$, $f(0) = f(1) = 0$. Then $f(x)$ has the form

$$f(x) = \begin{cases} Ax + B & 0 \leq x \leq a; \\ Cx + D & a \leq x \leq 1. \end{cases}$$

The boundary conditions mean that $B = 0$ and $D = -C$. For continuity at a we need

$$Aa = C(a - 1),$$

and for a jump of 1 in $f'(x)$ at $x = a$ we need $A + 1 = C$. Hence

$$A = a - 1, \quad B = 0, \quad C = a, \quad D = -a,$$

and

$$f(x) = \begin{cases} (a - 1)x & 0 \leq x \leq a; \\ a(x - 1) & a \leq x \leq 1. \end{cases}$$