Solution (#1722) Consider the DE

$$f''(x)+\omega^2 f(x)=k(x) \qquad \text{where} \qquad f(0)=f'(0)=0.$$

Applying the Laplace transform we obtain

$$s^2\bar{f}(s)+\omega^2\bar{f}(s)=\bar{k}(s),$$

so

$$\bar{f}(s)=\frac{\bar{k}(s)}{s^2+\omega^2}.$$

As $\bar{k}(s)$ is the transform of $k(x)$, and $\left(s^2+\omega^2\right)^{-1}$ is the transform of $\omega^{-1}\sin\omega x$, then by the convolution theorem

$$f(x)=\frac{k(x)*\sin(\omega x)}{\omega}.$$

If $k(x)=\delta(x-a)$ where $a>0$ we have by the sifting property that

$$f(x)=\frac{1}{\omega}\int_0^x \delta(t-a)\sin\omega(x-t)\,\mathrm{d}t=\left\{\begin{array}{cc} 0 & a>x \\ \omega^{-1}\sin\omega(x-a) & a\leqslant x \end{array}\right. .$$