Solution (#1723) Let k be real and $a > 0$. Consider the DE

$$f''(x) - 3f'(x) + 2f(x) = k\,\delta(x-a), \qquad f(0) = f\,'(0) = 1.$$

Working directly we see $f(x)$ has the form

$$f(x) = \begin{cases} Ae^x + Be^{2x} & x \leqslant a; \\ Ce^x + De^{2x} & a \leqslant x. \end{cases}$$

The initial conditions give us

$$A + B = 1 = A + 2B.$$

So $A = 1$ and $B = 0$. Continuity at a means

$$e^a = Ce^a + De^{2a},$$

and a jump of k in $f'(x)$ at $x = a$ means

$$e^a + k = Ce^a + 2De^{2a}.$$

Hence

$$A = 1, \qquad B = 0, \qquad C = 1 - ke^{-a}, \qquad D = ke^{-2a}$$

and

$$f(x) = \begin{cases} e^x & x \leqslant a; \\ e^x - ke^{x-a} + ke^{2(x-a)} & a \leqslant x. \end{cases}$$

Alternatively if we had used the Laplace transform, then we find

$$\left\{s^2\bar{f} - s - 1\right\} - 3\left\{s\bar{f} - 1\right\} + 2\bar{f} = ke^{-as}.$$

Rearranging we have

$$(s-1)\,(s-2)\,\bar{f} = s - 2 + ke^{-as}$$

and hence

$$\bar{f} = \frac{1}{s-1} + \frac{ke^{-as}}{(s-1)(s-2)} = \frac{1}{s-1} + ke^{-as}\left\{\frac{1}{s-2} - \frac{1}{s-1}\right\}.$$

The Laplace inverse of $(s-2)^{-1} - (s-1)^{-1}$ is $e^{2x} - e^x$ and hence

$$f(x) = e^x + k\left(e^{2(x-a)} - e^{x-a}\right)H(x-a).$$