**Solution** (#1737) The planet's position vector $\mathbf{r}(t)$ satisfies

$$\ddot{\mathbf{r}} = -\frac{GM}{r^2}\mathbf{e},$$

where $M$ is the mass of the sun and $G$ is the universal gravitational constant.
(i) By taking components of the acceleration in the $\mathbf{e}$ and $\mathbf{f}$ directions we find

$$\ddot{r} - r\dot{\theta}^2 = \frac{-GM}{r^2} \qquad \text{and} \qquad \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$$

and so $r^2\dot{\theta} = h$ is constant during the planet's motion.
(ii) Set $u = 1/r$ so that $r = 1/u$. We also have $h = r^2\dot{\theta}$. So by the chain rule

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{1}{r^2\dot{\theta}}\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\dot{r}}{h} \qquad \Longrightarrow \qquad \dot{r} = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}.$$

Similarly

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\dot{r}}{h}\right) = -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{\ddot{r}}{h}\frac{r^2}{h} \qquad \Longrightarrow \qquad \ddot{r} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}.$$

(iii) Recall, taking the component of the acceleration parallel to $\mathbf{e}$, we found

$$\ddot{r} - r\dot{\theta}^2 = \frac{-GM}{r^2}.$$

As $\dot{\theta} = h/r^2 = hu^2$ and using (ii) we find

$$-h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - \frac{1}{u}\left(hu^2\right)^2 = -GMu^2$$

which simplifies as

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{GM}{h^2}.$$

This has general solution

$$u(\theta) = A\cos\theta + B\sin\theta + \frac{GM}{h^2}.$$

By changing the choice of half-line $\theta = 0$ appropriately we may assume that $B = 0$ and $A > 0$. Alternatively we can employ an identity $A\cos\theta + B\sin\theta = R\cos(\theta - \alpha)$. We now have

$$\frac{1}{r(\theta)} = A\cos\theta + \frac{GM}{h^2},$$

which we recognize to be a conic section from (4.18).
(iv) Suppose initially that $\theta = 0$, $\mathbf{r} = (R, 0)$ and $\dot{\mathbf{r}} = (0, V)$. We then have initially from the velocity components found in (ii) that

$$r = R, \qquad \dot{r} = 0, \qquad r\dot{\theta} = V.$$

So $h = r(r\dot{\theta}) = RV$ and

$$\frac{1}{R} = A + \frac{GM}{R^2V^2}.$$

Hence

$$\frac{1}{r(\theta)} = \left(\frac{1}{R} - \frac{GM}{R^2V^2}\right)\cos\theta + \frac{GM}{R^2V^2}.$$

The planet's path (a conic) will be an ellipse if $r(\theta)$ remains bounded and so we need that the RHS never reaches zero. For this to happen we need

$$\frac{1}{R} - \frac{GM}{R^2V^2} < \frac{GM}{R^2V^2},$$

or equivalently

$$RV^2 < 2GM.$$