Solution (#1741) Let A be a square matrix, and $M(t)$ be matrices such that $M'(t) = AM(t)$ and $M(0) = I$.

By assumption we know that there are matrices e^{At} such that

$$\text{(i)} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{At}\right) = Ae^{At}, \qquad \text{(ii)} \quad e^{At} = I \text{ when } t = 0, \quad \text{(iii)} \qquad e^{At} \text{ commutes with } A.$$

If we consider $N(t) = e^{-At}M(t)$ then we have that $N(0) = I^2 = I$ and also by the product rule that

$$\begin{aligned} N'(t) &= -Ae^{-At}M(t) + e^{-At}M'(t) \\ &= -Ae^{-At}M(t) + e^{-At}AM(t) \qquad \text{[by (i)]} \\ &= -Ae^{-At}M(t) + Ae^{-At}M(t) \qquad \text{[by (iii)]} \\ &= 0. \end{aligned}$$

It follows that $N(t)$ is constant and so that

$$e^{-At}M(t) = I \qquad \text{for all } t.$$

As e^{At} has the necessary properties of $M(t)$ then we have

$$e^{-At}e^{At} = I$$

and so these matrices are inverses of one another. Hence

$$M(t) = e^{At}e^{-At}M(t) = e^{At},$$

as required.