Solution (#1748) Let

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where a, b, c are real and $a \neq c$. In a similar fashion to #1747 we set

$$M(t) = e^{At} = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}$$

with

$$M'(t) = AM(t), \qquad M(0) = I_2.$$

So

$$\alpha'(t) = a\alpha(t) + b\gamma(t), \qquad \beta'(t) = a\beta(t) + b\delta(t), \qquad \gamma'(t) = c\gamma(t), \qquad \delta'(t) = c\delta(t).$$

Looking at the last two equations we see that

$$\gamma(t) = \gamma(0)e^{ct} = 0; \qquad \delta(t) = \delta(0)e^{ct} = e^{ct}.$$

Then

$$\alpha(t) = \alpha(0)e^{at} = e^{at}.$$

Finally

$$\beta'(t) - a\beta(t) = be^{ct}.$$

Applying an integrating factor of e^{-at} and integrating we see

$$\beta(t)e^{-at} = \frac{b}{c-a}e^{(c-a)t} + \text{const.}.$$

As $\beta(0) = 0$ then

$$\beta(t)e^{-at} = \frac{b}{c-a}e^{(c-a)t} - \frac{b}{c-a} \qquad \Longrightarrow \qquad \beta(t) = \frac{b(e^{ct} - e^{at})}{c-a}.$$

Hence setting $t = 1$ we have

$$e^A = \begin{pmatrix} e^a & b\left(\frac{e^a - e^c}{a-c}\right) \\ 0 & e^c \end{pmatrix} \qquad \text{when } a \neq c.$$

If $a = c$ we can make a similar argument to the one above. Alternatively we can let a tend to c noting that

$$\lim_{a \to c} \frac{e^a - e^c}{a - c} = e^c$$

as this limit is the derivative of e^x at $x = c$. Hence

$$\exp\left(\begin{pmatrix} c & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} e^c & be^c \\ 0 & e^c \end{pmatrix}.$$