Solution (#1750) Let A be a square matrix such that $A^k = 0$. If we set

$$M(t) = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots + \frac{A^{k-1}}{(k-1)!}t^{k-1}$$

then we see that $M(0) = I$ and also that

$$\begin{aligned}
M'(t) &= A + \frac{A^2}{2!}2t + \frac{A^3}{3!}3t^2 + \cdots + \frac{A^{k-1}}{(k-1)!}(k-1)t^{k-2} \\
&= A + A^2 t + \frac{A^3}{2!}t^2 + \cdots + \frac{A^{k-1}}{(k-2)!}t^{k-2} \\
&= A\left[I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^{k-2}}{(k-2)!}t^{k-2}\right] \\
&= A\left[I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^{k-2}}{(k-2)!}t^{k-2}\right] + \frac{A^k}{(k-1)!}t^{k-1} \qquad [\text{as } A^k = 0] \\
&= A\left[I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^{k-2}}{(k-2)!}t^{k-2} + \frac{A^{k-1}}{(k-1)!}t^{k-1}\right] \\
&= AM(t).
\end{aligned}$$

It follows that $M(t) = e^{At}$ and

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^{k-1}}{(k-1)!}.$$