Solution (#1753) Let A be a square matrix such that $A^2 = 2A - I$. Say

$$e^{At} = a_0(t)I + a_1(t)A.$$

From Remark 6.81(c) we have

$$a_1''(t) - 2a_1'(t) + a_1(t) = 0.$$

The auxiliary equation has repeated roots $1, 1$ and so the DE has general solution

$$a_1(t) = (Ct + D)e^t.$$

As $e^{At} = I$ when $t = 0$ then

$$a_1(0) = 0 \qquad \text{and} \qquad a_1'(0) = a_0(0) + 2a_1(0) = 1.$$

Hence $D = 0$ and $C = 1$, giving

$$a_1(t) = te^t.$$

Then

$$a_0(t) = a_1'(t) - 2a_1(t) = (t+1)e^t - 2te^t = (1-t)e^t.$$

So

$$e^{At} = (1-t)e^t I + te^t A.$$