Solution (#1757) Let

$$A=\begin{pmatrix}1&2&-1&3\\0&1&1&-2\\0&0&1&1\\0&0&0&1\end{pmatrix}\qquad\text{and}\qquad B=\begin{pmatrix}2&0&0&0\\0&1&2&-1\\0&0&1&3\\0&0&0&1\end{pmatrix}.$$

Note that $(A-I)^4=0$ and so by #1750 we have

$$e^{A-I}=I+(A-I)+\frac{1}{2!}(A-I)^2+\frac{1}{3!}(A-I)^3.$$

Now

$$A-I=\begin{pmatrix}0&2&-1&3\\0&0&1&-2\\0&0&0&1\\0&0&0&0\end{pmatrix};\qquad(A-I)^2=\begin{pmatrix}0&0&2&-5\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix};\qquad(A-I)^3=\begin{pmatrix}0&0&0&2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

so that

$$e^{A-I}=\begin{pmatrix}1&2&0&\frac{5}{6}\\0&1&1&-\frac{3}{2}\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

Finally we have $e^A=e^{I+(A-I)}=e^Ie^{A-I}=(eI)\,e^{A-I}$ and so

$$e^A=e\begin{pmatrix}1&2&0&\frac{5}{6}\\0&1&1&-\frac{3}{2}\\0&0&1&1\\0&0&0&1\end{pmatrix}.$$

We can write $B=\operatorname{diag}(2I_1,C)$ where C is a 3×3 matrix such that $(C-I_3)^3=0$. So by #1750 again we have

$$e^{C-I}=I+(C-I)+\frac{1}{2!}(C-I)^2.$$

Now

$$C-I=\begin{pmatrix}0&2&-1\\0&0&3\\0&0&0\end{pmatrix};\qquad(C-I)^2=\begin{pmatrix}0&0&6\\0&0&0\\0&0&0\end{pmatrix},$$

so that

$$e^{C-I}=\begin{pmatrix}1&2&2\\0&1&3\\0&0&1\end{pmatrix}.$$

Finally by #1744(ii) and as $e^C=e^Ie^{C-I}=ee^{C-I}$ we have

$$e^B=\begin{pmatrix}e^2&0&0&0\\0&e&2e&2e\\0&0&e&3e\\0&0&0&e\end{pmatrix}.$$