Solution (#249) (i) Note that

$$\begin{aligned} A_2 \geqslant G_2 \quad &\Longleftrightarrow \quad \frac{x_1+x_2}{2} \geqslant \sqrt{x_1 x_2} \\ &\Longleftrightarrow \quad (x_1+x_2)^2 \geqslant 4x_1x_2 \\ &\Longleftrightarrow \quad x_1^2 - 2x_1x_2 + x_2^2 \geqslant 0 \\ &\Longleftrightarrow \quad (x_1-x_2)^2 \geqslant 0. \end{aligned}$$

This holds true for all x_1, x_2 and we have equality if and only if $x_1 = x_2$.

(ii) Let $x_1, \ldots, x_{n+1}$ be $n+1$ positive numbers. Set

$$\mu = \frac{x_1 + x_2 + \cdots + x_{n+1}}{n+1}$$

and assume that $x_n < \mu < x_{n+1}$. Further set

$$X_1 = x_n + x_{n+1} - \mu, \qquad X_2 = \mu$$

(noting X_1 and X_2 are positive) and note that

$$X_1 X_2 = (x_n + x_{n+1} - \mu)\mu = (\mu - x_n)(x_{n+1} - \mu) + x_n x_{n+1} > x_n x_{n+1}.$$

The remainder of the proof follows by induction. Suppose that $A_n \geqslant G_n$ for any n positive numbers with equality if and only if all the numbers are equal. This holds for $n = 2$.

Let $x_1, \ldots, x_{n+1}$ be $n+1$ positive numbers. If they are all equal then $A_{n+1} = G_{n+1}$. If not, by relabelling the numbers if necessary, we may assume $x_n < \mu < x_{n+1}$ where $\mu = A_{n+1}$.
We now apply the assumed inequality to the n numbers $x_1, x_2, \ldots, x_{n-1}, X_1$. Note that these have arithmetic mean

$$\frac{x_1 + x_2 + \cdots + x_{n-1} + X_1}{n} = \frac{x_1 + x_2 + \cdots + x_n - \mu}{n} = \frac{(n+1)\mu - \mu}{n} = \mu,$$

and this must at least equal their geometric mean by hypothesis. Hence

$$\mu^{n+1} = \mu^n \times X_2 \geqslant (x_1 x_2 \cdots x_{n-1} X_1)\, X_2 > x_1 x_2 \cdots x_{n-1} x_n x_{n+1}.$$

The result follows by induction.