Solution (#251) Let $x_1, x_2, \ldots, x_n$ be positive numbers. The means A_n and Q_n are defined by

$$A_n = \frac{x_1 + \cdots + x_n}{n}, \qquad Q_n = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

So we have

$$\begin{aligned} Q_n \geqslant A_n \quad &\iff \quad \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geqslant \frac{x_1 + \cdots + x_n}{n} \\ &\iff \quad n(x_1^2 + \cdots + x_n^2) \geqslant (x_1 + \cdots + x_n)^2 \\ &\iff \quad n \sum x_i^2 \geqslant \sum x_i^2 + 2 \sum_{i<j} x_i x_j \\ &\iff \quad \sum_{i<j} (x_i - x_j)^2 \geqslant 0. \end{aligned}$$

Hence $Q_n \geqslant A_n$ and we have equality if and only if all the x_i are equal.