Solution (#253) (i) By an inductive argument and using

$$(1+\sqrt{2})^{n+1} = (1+\sqrt{2})(1+\sqrt{2})^n$$

we can show that

$$(1+\sqrt{2})^n = a_n + b_n\sqrt{2},$$

where a_n, b_n are integers. Further by the irrationality of $\sqrt{2}$ these integers are unique. We find

$$a_{n+1} = a_n + 2b_n, \qquad b_{n+1} = a_n + b_n.$$

We see by induction that a_n and b_n are integers for all n. If we had

$$(1+\sqrt{2})^n = a_n + b_n\sqrt{2} = c_n + d_n\sqrt{2}$$

for integers a_n, b_n, c_n, d_n then we can again conclude that $a_n = c_n$, $b_n = d_n$ as $\sqrt{2}$ is irrational.

(ii) This follows from showing

$$(a_{n+1})^2 - 2\,(b_{n+1})^2 = -\left(a_n^2 - 2b_n^2\right).$$

(iii) Note that $\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right) = 1$ and so we may write

$$(1+\sqrt{2})^{-n} = a_{-n} + b_{-n}\sqrt{2}$$

where a_{-n} and b_{-n} are integers. We ultimately find

$$a_{-n} = (-1)^n\, a_n, \qquad b_{-n} = (-1)^{n+1}\, b_n.$$