Solution (#257) When $n = 1$ then

$$LHS = \sin x; \qquad RHS = \frac{\sin x \sin \frac{x}{2}}{\sin \frac{x}{2}} = \sin x$$

as required. If

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\left\{\frac{1}{2}(n+1)x\right\}\sin\left\{\frac{1}{2}nx\right\}}{\sin\left\{\frac{1}{2}x\right\}} \tag{9.7}$$

is true for $n = N$ then

$$\sum_{k=1}^{N+1} \sin kx = \sum_{k=1}^{N} \sin kx + \sin(N+1)x = \frac{\sin\frac{(N+1)x}{2}\sin\frac{Nx}{2}}{\sin\frac{x}{2}} + \sin(N+1)x.$$

Using the identity

$$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right),$$

we have

$$\sin\frac{(N+2)x}{2} - \sin\frac{Nx}{2} = 2\cos\frac{(N+1)x}{2}\sin\frac{x}{2}.$$

Hence

$$\begin{aligned}
\sum_{k=1}^{N+1} \sin kx &= \frac{\sin\frac{(N+1)x}{2}\left[\sin\frac{(N+2)x}{2} - 2\cos\frac{(N+1)x}{2}\sin\frac{x}{2}\right]}{\sin\frac{x}{2}} + \sin(N+1)x \\
&= \frac{\sin\frac{(N+1)x}{2}\sin\frac{(N+2)x}{2}}{\sin\frac{x}{2}} - 2\sin\frac{(N+1)x}{2}\cos\frac{(N+1)x}{2} + \sin(N+1)x \\
&= \frac{\sin\frac{(N+1)x}{2}\sin\frac{(N+2)x}{2}}{\sin\frac{x}{2}},
\end{aligned}$$

as required to complete the proof by induction.