Solution (#278) When $n=1$ then the LHS and RHS of

$$\binom{2n}{n} \leqslant \frac{2^{2n}}{\sqrt{3n+1}} \tag{9.8}$$

are both 2. Suppose that (9.8) is true in a particular nth case. Then

$$\binom{2n+2}{n+1} = \frac{(2n+2)(2n+1)}{(n+1)^2}\binom{2n}{n} \leqslant \binom{2n+1}{n+1}\frac{2^{2n+1}}{\sqrt{3n+1}}.$$

In order to complete the inductive step we now need to show

$$\binom{2n+1}{n+1}\frac{2^{2n+1}}{\sqrt{3n+1}} \leqslant \frac{2^{2n+2}}{\sqrt{3n+4}}. \tag{9.9}$$

For $n \geqslant 1$ we have

$$\begin{aligned} 12n^3+28n^2+19n+4 &\leqslant 12n^3+2n^2+20n+4, \\ (2n+1)^2(3n+4) &\leqslant 4(n+1)^2(3n+1), \\ \binom{2n+1}{n+1}\frac{2^{2n+1}}{\sqrt{3n+1}} &\leqslant \frac{2^{2n+2}}{\sqrt{3n+4}}, \end{aligned}$$

proving (9.9) and we conclude

$$\binom{2n+2}{n+1} \leqslant \frac{2^{2n+2}}{\sqrt{3n+4}}.$$

The proof of the right inequality follows by induction. The other inequality follows in a similar fashion.

When $n=1$ then the LHS and RHS of

$$\binom{2n}{n} \geqslant \frac{2^{2n}}{2\sqrt{n}} \tag{9.10}$$

are both 2. Suppose now that (9.10) is true in a particular nth case. Then

$$\binom{2n+2}{n+1} = \frac{(2n+2)(2n+1)}{(n+1)^2}\binom{2n}{n} \geqslant \binom{2n+1}{n+1}\frac{2^{2n+1}}{2\sqrt{4n}}.$$

For $n \geqslant 1$ we have

$$\begin{aligned} (2n+1)^2 &\geqslant 4(n+1)n \\ \binom{2n+1}{n+1}\frac{1}{2\sqrt{n}} &\geqslant \frac{2}{2\sqrt{n+1}} \\ \binom{2n+1}{n+1}\frac{2^{2n+1}}{2\sqrt{n}} &\geqslant \frac{2^{2n+2}}{2\sqrt{n+1}}, \end{aligned}$$

showing

$$\binom{2n+2}{n+1} \geqslant \binom{2n+1}{n+1}\frac{2^{2n+1}}{2\sqrt{n}} \geqslant \frac{2^{2n+2}}{2\sqrt{n+1}}$$

and the proof of the left inequality follows by induction.