Solution (#280) Let $n \geqslant 2$ be an integer. Denote the finite product

$$p_n = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\times\cdots\times\left(1-\frac{1}{n^2}\right).$$

If we factorize the brackets as

$$1-\frac{1}{k^2} = \frac{k^2-1}{k^2} = \frac{(k-1)(k+1)}{k\times k}$$

then we see that all the internal factors in the product cancel out save for the first $\frac{1}{2}$ and last $\frac{n+1}{n}$ so that p_n equals

$$\frac{1\times 3}{2\times 2}\times\frac{2\times 4}{3\times 3}\times\frac{3\times 5}{4\times 4}\times\cdots\times\frac{(n-2)\times n}{(n-1)\times(n-1)}\times\frac{(n-1)\times(n+1)}{n\times n} = \frac{1}{2}\times\frac{n+1}{n} = \frac{n+1}{2n}.$$

As n tends to infinity this approaches $1/2$ and so the infinite product equals

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\cdots = \frac{1}{2}.$$

If we now just focus on the even terms then we note (using the same factorization of the terms as above) that

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{36}\right)\times\cdots = \frac{1\times 3}{2\times 2}\times\frac{3\times 5}{4\times 4}\times\frac{5\times 7}{6\times 6}\times\cdots = \frac{2}{\pi},$$

by #279. Hence

$$\begin{aligned} & \left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\left(1-\frac{1}{81}\right)\cdots \\ = & \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)}{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{36}\right)\left(1-\frac{1}{64}\right)} \\ = & \frac{1/2}{2/\pi} = \frac{\pi}{4}. \end{aligned}$$