Solution (#300) Suppose now that a fair coin is tossed $2n+1$ times with a player winning after $n+1$ favourable tosses. Suppose there have been h heads and t tails before the game is interrupted where $h, t < n+1$. Let $\tilde{h} = n+1-h$ and $\tilde{t} = n+1-t$. The game will be won in the next $\tilde{h}+\tilde{t}-1$ tosses and we need to consider which of these $2^{\tilde{h}+\tilde{t}-1}$ combinations lead to a heads/tails win.

The game can be won with heads on the $(\tilde{h}+k)$th toss where $0 \leqslant k \leqslant \tilde{t}-1$. The number of combinations (from the $2^{\tilde{h}+\tilde{t}-1}$ combinations mentioned above) that lead to this is

$$\binom{\tilde{h}+k-1}{\tilde{h}-1} 2^{\tilde{t}-1-k}$$

as the $(\tilde{h}+k)$th toss must be heads with $\tilde{h}-1$ heads coming previously.

So the probability of the player needing heads winning is

$$\frac{1}{2^{\tilde{h}+\tilde{t}-1}} \sum_{k=0}^{\tilde{t}-1} \binom{\tilde{h}+k-1}{\tilde{h}-1} 2^{\tilde{t}-1-k} = \frac{1}{2^{\tilde{h}}} \sum_{k=0}^{\tilde{t}-1} \binom{\tilde{h}+k-1}{\tilde{h}-1} 2^{-k}$$

and by symmetry the probability of the player needing tails winning is

$$\frac{1}{2^{\tilde{t}}} \sum_{k=0}^{\tilde{h}-1} \binom{\tilde{t}+k-1}{\tilde{t}-1} 2^{-k}.$$

If $h = n$, so that only one further head is needed, we have $\tilde{h} = 1$. Hence the sum giving the probability of a heads win is

$$\frac{1}{2} \sum_{k=0}^{\tilde{t}-1} 2^{-k} = \sum_{k=1}^{\tilde{t}} 2^{-k} = 1 - 2^{-\tilde{t}}$$

This is as expected because a tails win can only occur with an immediate run of $\tilde{t}$ tails. And when $t = n$ so that $\tilde{t} = 1$ we obtain

$$\frac{1}{2^{\tilde{h}}} \sum_{k=0}^{0} \binom{\tilde{h}+k-1}{\tilde{h}-1} 2^{-k} = \frac{1}{2^{\tilde{h}}},$$

which is again as expected as heads can only win with an immediate run of $\tilde{h}$ heads.