Solution (#309) Let n be a natural number and let $\omega = \text{cis}\,(2\pi/3)$ be a cube root of unity so that $\omega^3 = 1$.

(i) If $k = 3m$ then

$$1 + \omega^k + \omega^{2k} = 1 + \omega^{3m} + \omega^{6m} = 1 + \left(\omega^3\right)^m + \left(\omega^3\right)^{2m} = 1 + 1 + 1 = 3.$$

If $k = 3m + 1$ then

$$1 + \omega^k + \omega^{2k} = 1 + \omega^{3m+1} + \omega^{6m+2} = 1 + \left(\omega^3\right)^m \omega + \left(\omega^3\right)^{2m} \omega^2 = 1 + \omega + \omega^2 = 0.$$

If $k = 3m + 2$ then

$$1 + \omega^k + \omega^{2k} = 1 + \omega^{3m+2} + \omega^{6m+4} = 1 + \left(\omega^3\right)^m \omega^2 + \left(\omega^3\right)^{2m+1} \omega = 1 + \omega^2 + \omega = 0.$$

(ii) Hence

$$\begin{aligned} & \frac{(1+x)^n + (1+\omega x)^n + \left(1+\omega^2 x\right)^n}{3} \\ = \; & \frac{1}{3} \sum_{k=0}^{n} \binom{n}{k} \left(1 + \omega^k + \omega^{2k}\right) x^k \\ = \; & \binom{n}{0} + \binom{n}{3} x^3 + \binom{n}{6} x^6 + \binom{n}{9} x^9 + \cdots \end{aligned}$$

Setting $x = 1$ we have

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \binom{n}{9} + \cdots = \frac{2^n + (1+\omega)^n + \left(1+\omega^2\right)^n}{3}.$$

Now $\omega^2 = \bar{\omega}$ and so $(1+\omega)^n + \left(1+\omega^2\right)^n = 2\,\text{Re}\left\{(1+\omega)^n\right\}$. Further

$$1 + \omega = -\omega^2 = \text{cis}\,(\pi/3)$$

Finally then by De Moivre's theorem

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \binom{n}{9} + \cdots = \frac{2^n + 2\,\text{Re}\left(\text{cis}\,(\pi/3)\right)^n}{3} = \frac{2^n + 2\cos\left(n\pi/3\right)}{3}$$