Solution (#319) Let $n \geqslant 0$. In #75 we set $z = \operatorname{cis}\theta$ and saw $2\cos n\theta = z^n + z^{-n}$. So by the binomial theorem we have that

$$\begin{aligned}
2^{2n+1}\cos^{2n+1}\theta &= \left(z+z^{-1}\right)^{2n+1} \\
&= \sum_{k=0}^{2n+1}\binom{2n+1}{k} z^k \left(z^{-1}\right)^{2n+1-k} \\
&= \sum_{k=0}^{2n+1}\binom{2n+1}{k} z^{2k-2n-1} \\
&= \sum_{k=0}^{n}\binom{2n+1}{k}\left(z^{2k-2n-1}+z^{2n+1-2k}\right) \\
&= 2\sum_{k=0}^{n}\binom{2n+1}{k}\cos\left[(2n-2k+1)\theta\right]
\end{aligned}$$

where the penultimate sum is arrived at by combining the kth and $(2n+1-k)$th terms for $0 \leqslant k < n$.

If we integrate both sides with respect to θ between the limits of $\theta = 0$ and $\theta = \pi/2$ and use the formula found in #263 we see

$$\begin{aligned}
\frac{2^{4n+1}(n!)^2}{(2n+1)!} &= 2^{2n+1}\int_0^{\pi/2}\cos^{2n+1}\theta\,\mathrm{d}\theta \\
&= 2\sum_{k=0}^{n}\binom{2n+1}{k}\int_0^{\pi/2}\cos\left[(2n-2k+1)\theta\right]\,\mathrm{d}\theta \\
&= 2\sum_{k=0}^{n}\binom{2n+1}{k}\left[\frac{\sin(2n-2k+1)\theta}{2n-2k+1}\right]_0^{\pi/2} \\
&= 2\sum_{k=0}^{n}\frac{1}{2n-2k+1}\binom{2n+1}{k}\sin(n-k+1/2)\pi \\
&= 2\sum_{k=0}^{n}\frac{(-1)^{n-k}}{2n-2k+1}\binom{2n+1}{k}.
\end{aligned}$$

Rearranging we then arrive at

$$(-1)^n\frac{2^{4n}(n!)^2}{(2n+1)!} = \sum_{k=0}^{n}\frac{(-1)^k}{2n-2k+1}\binom{2n+1}{k}.$$